Single Server Retrial Queuing System with Second Optional Service under Coxian Phase Type Services

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ABSTRACT

Consider a single server retrial queueing system with second optional service under Coxian phase type services in which customers arrive in a Poisson process with arrival rate λ. Let k be the number of phases in the service station. The server provides two types of services namely Regular Service and Second Optional Service. The regular service time follows an exponential distribution with parameter μ_j for j^{th} phase ($j = 1, 2, 3, \ldots, k$). The second optional service time follows an exponential distribution with parameter μ. The services in all phases are independent and only one customer at a time is in the service mechanism. Let q_j ($j = 1,2,3,\ldots, k-1$) be the probability that the customer moves from j^{th} phase to $(j+1)^{th}$ phase. If the server is free at the time of a primary call arrival, the arriving call begins to be served in Phase 1 immediately by the server. If the server is busy, then the arriving customer goes to orbit and becomes a source of repeated calls. We assume that the access from orbit to the service facility is governed by the classical retrial policy. This model is solved by using Direct Truncation Method. Numerical studies have been done for analysis of mean number of customers in the orbit (MNCO),Truncation level (OCUT), Probabilities of server free and busy and for various values of $\lambda, q_1, q_2, \ldots, q_{k-1}, \mu_1, \mu_2, \mu_3, \ldots, \mu_k, \mu, p, k$ and σ and also various particular cases of this model have been discussed.

1. INTRODUCTION

Queueing systems, in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time, are called Retrial queues. For detailed survey of retrial queues and bibliographical information see Artalejo **[1, 2, 3],** Artalejo J.R and A. Gomez-Corral **[4],** Falin **[7],** monograph by Falin and Templeton **[8],** Yang and Templeton **[15]**. Because of the complexity of the retrial queueing models, analytic results are generally difficult to obtain. There are a great number of numerical and approximations methods available. In this paper we restrict our attention to solutions by Direct Truncation method **[5, 6, 10].**

Second optional service plays a vital role in retrial queueing systems. Kailash C. Madan [**11**] has studied an M/G/1 queue with second optional service using supplementary variable technique. The work of Madan was generalized by Medhi **[12]** who studied a single server Poisson input queue with a second optional channel. Gaudham Choudhury **[9]** studied some aspects of M/G/1 queueing system with second optional service and derived the steady state queue size distribution at the stationary point of time for general second optional service. Wang **[14]** studied M/G/1 queue with second optional service and server breakdown. Ayyappan *et al.*, **[6]** have studied second optional service for Retrial queueing system with priority services. Gopal sekar *et al*.,**[10]** have studied the behaviour of Retrial queueing system with second optional service under Erlang-k services. In this paper, we introduce the concept of second optional service for Coxian phase type service.

In day to day life, we face many queueing situations in which customers require the essential service and only some may require the additional service provided by the server. The following examples give us motivation to develop this model.

- In a barber shop, everyone may need a hair-cut (essential service) but only a few of the customers may need a colouring of hair (optional service).

- Students joining a Mathematics department of a university want to complete their post graduate program (essential service) but only some of them may join as Research Assistant (second optional service) soon after completing the postgraduate program.

- All patients who come to meet a doctor for curing their disease (essential service) but only some of them require Medical certificate (optional service).

2. MODEL DESCRIPTION

Consider a single server retrial queueing system with second optional service under Coxian phase type services in which customers arrive in a Poisson process with arrival rate $\boldsymbol{\lambda}$. These customers are identified as primary calls**.** Let k be the number of phases in the service station. The server provides two types of services namely Regular Service and Second Optional Service. The regular service time follows an exponential distribution with service rate μ_j for j^{th} phase (j = 1, 2, 3, . . . , k) and the second optional service time follows an exponential distribution with parameter μ. We assume that the services in all phases are independent and only one customer at a time is in the service mechanism. Let q_j (j = 1,2,3,. . . , k-1) be the probability that the customer moves from j^{th} phase to $(j+1)^{th}$ phase. If the server is free at the time of a primary call arrival, the arriving call begins to be served in phase 1 immediately by the server. The sequence of phases could be arranged one after the other in series formation, with the provision of termination after the completion of any phase; that is the customer may terminate from j^{th} phase with probability $(1-q_j)$ and then opt for the optional service with probability p and leaves the system after completion of the second optional service, after which the next customer enters the first phase unless he declines the optional service (with probability (1-p)) and leaves the system, after which the next customer enters the first phase. If the server is busy, then the arriving customer goes to orbit and becomes a source of repeated calls. This pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity $\boldsymbol{\sigma}$. If an incoming repeated call finds the server free, it is served in the same manner and leaves the system after service, while the source which produced this repeated call disappears. Otherwise, the system state does not change.

2.1 Retrial Policy

We assume that the access from the orbit to the service facility follows an exponential distribution with rate $n\sigma$ which may depend on the current number n, ($n \geq 0$) the number of customers in the orbit. That is, the probability of repeated attempts during the interval (t, $t + \Delta t$), given that there are n customers in the orbit at time t is $n\sigma\, \Delta t$. It is called the classical retrial policy. The input flow of primary calls, interval between repetitions and service times in phases are mutually independent.

2.2 Description of Random Process

Let N(t) be the random variable which represents the number of customers in orbit at time t and S(t) be the random variable which represents the server status at time t. The random process is described as

$X = \{ < N(t) , S(t) > / N(t) = 0, 1, 2, 3,. . . ; S(t) = 0, 1, 2, 3,. . . , k+1\}$

$S(t) = 0$; the server being idle

$S(t) = j$; the server being busy with the customer in the j^{th} phase.

$S(t) = k+1$; the server being busy with a customer who opted for optional service.

The possible state space is

$$\{ (i , j) / i = 0,1,2,3,\ldots ; \; j = 0,1,2,3, . . . , k+1 \}$$

The infinitesimal generator matrix **Q** for this model is given below

$$\mathbf{Q} = \begin{pmatrix} A_{00} & A_0 & O & O & O & \ldots \\ A_{10} & A_{11} & A_0 & O & O & \ldots \\ O & A_{21} & A_{22} & A_0 & O & \ldots \\ O & O & A_{32} & A_{33} & A_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

The matrices described in the Infinitesimal generator matrix Q can be obtained from the following infinitesimal transition rates of process X as follows

$$q_{(0,j)(l,m)} = \begin{cases} \lambda & \text{if } (l, m) = (0,1) \; ; \; j = 0 \\ -\lambda & \text{if } (l, m) = (0,0) \; ; \; j = 0 \\ q_j \mu_j & \text{if } (l, m) = (0, j+1) \; ; \; 1 \leq j \leq k-1 \\ \lambda & \text{if } (l, m) = (1, j) \; ; \; 1 \leq j \leq k+1 \\ p(1-q_j)\mu_j & \text{if } (l, m) = (0, k+1) \; ; \; 1 \leq j \leq k-1 \\ (1-p)(1-q_j)\mu_j & \text{if } (l, m) = (0, 0) \; ; \; 1 \leq j \leq k-1 \\ p\mu_k & \text{if } (l, m) = (0,k+1) \; ; \; j = k \\ (1-p)\mu_k & \text{if } (l, m) = (0,0) \; ; \; j = k \end{cases}$$

μ if $(l, m) = (0,0)$; $j = k+1$
$-(\lambda+\mu)$ if $(l, m) = (0, j)$; $j = k+1$
$-(\lambda+\mu_j)$ if $(l, m) = (0, j)$; $1 \le j \le k$
0 otherwise

For i =1, 2, 3. . .

$$q_{(i,j)(l,m)} = \begin{cases} i\sigma & \text{if } (l, m) = (i-1,1) \;;\; j = 0 \\ \lambda & \text{if } (l, m) = (i,1) \;;\; j = 0 \\ -(\lambda + i\sigma) & \text{if } (l, m) = (i,0) \;;\; j = 0 \\ q_j \mu_j & \text{if } (l, m) = (i, j+1) \;;\; 1 \le j \le k-1 \\ \lambda & \text{if } (l, m) = (i+1,j) \;;\; 1 \le j \le k+1 \\ p(1-q_j)\mu_j & \text{if } (l, m) = (i,k+1) \;;\; 1 \le j \le k-1 \\ (1-p)(1-q_j)\mu_j & \text{if } (l, m) = (i,0) \;;\; 1 \le j \le k-1 \\ p\mu_k & \text{if } (l, m) = (i,k+1) \;;\; j = k \\ (1-p)\mu_k & \text{if } (l, m) = (i,0) \;;\; j = k \\ \mu & \text{if } (l, m) = (i,0) \;;\; j = k+1 \\ -(\lambda+\mu) & \text{if } (l, m) = (i, j) \;;\; j = k+1 \\ -(\lambda+\mu_j) & \text{if } (l, m) = (i,j) \;;\; 1 \le j \le k \\ 0 & \text{otherwise} \end{cases}$$

If the capacity of the orbit is finite say i = M

$$q_{(i,j)(l,m)} = \begin{cases} i\sigma & \text{if } (l, m) = (i-1,1) \;;\; j = 0 \\ \lambda & \text{if } (l, m) = (i,1) \;;\; j = 0 \\ -(\lambda + i\sigma) & \text{if } (l, m) = (i,0) \;;\; j = 0 \\ q_j \mu_j & \text{if } (l, m) = (i, j+1) \;;\; 1 \le j \le k-1 \\ p(1-q_j)\mu_j & \text{if } (l, m) = (i,k+1) \;;\; 1 \le j \le k-1 \\ (1-p)(1-q_j)\mu_j & \text{if } (l, m) = (i,0) \;;\; 1 \le j \le k-1 \\ p\mu_k & \text{if } (l, m) = (i,k+1) \;;\; j = k \\ (1-p)\mu_k & \text{if } (l, m) = (i,0) \;;\; j = k \\ \mu & \text{if } (l, m) = (i,0) \;;\; j = k+1 \\ -\mu & \text{if } (l, m) = (i, j) \;;\; j = k+1 \\ -\mu_j & \text{if } (l, m) = (i,j) \;;\; 1 \le j \le k \\ 0 & \text{otherwise} \end{cases}$$

3. DESCRIPTION OF COMPUTATIONAL METHOD

Retrial queueing models can be solved computationally by the following techniques.

(a) Direct Truncation Method
(b) Generalized Truncation Method
(c) Truncation Method using Level Dependent Quasi Birth and Death Process (LDQBD)
(d) Matrix Geometric Approximation.

In this paper we only discuss the Direct Truncation Method for solving the above model since we can very comfortably find the orbit cut (M) by using computational method and beyond the orbit cut all probabilities will be zero.

3.1 DIRECT TRUNCATION METHOD

Let **X** be the steady-state probability vector of **Q,** partitioned as **X** = (x(0), x(1), x(2), **. . .**) and **X** satisfies

$$\mathbf{XQ} = \mathbf{0} \quad \text{and} \quad \mathbf{Xe} = \mathbf{1} \tag{1}$$

where x(i) = (P_{i0} , P_{i1} , P_{i2} , . . . , P_{ik+1}) ; i = 0, 1, 2, **. . .**

The above system of equations (1) can be solved by means of truncating the system of equations for sufficiently large value of the number of customers in the orbit, say M. That is, the orbit size is restricted to M such that any arriving customer finding the orbit full is considered lost. The value of M can be chosen so that the loss probability is small. Due to the intrinsic nature of the system, the only choice available for studying M is through algorithmic methods. While a number of approaches are available for determining the cut-off point, M, the one that seems to perform well is to increase M until the largest individual change in the elements of **X** for successive values is less than ε a predetermined infinitesimal value.

If M denotes the cut-off point or Truncation level, then the steady state probability vector $\mathbf{X}^{(\mathbf{M})}$ is partitioned as $\mathbf{X}^{(\mathbf{M})}$ = (x(0) , x(1), x(2) , **. . .** , x(M)), where $\mathbf{X}^{(\mathbf{M})}$ satisfies

$$\mathbf{X}^{(\mathbf{M})}\,\mathbf{Q}^{(\mathbf{M})} = \mathbf{0} \quad \text{and} \quad \mathbf{X}^{(\mathbf{M})}\,\mathbf{e} = 1 \tag{2}$$

and x(i) = (P_{i0} , P_{i1} , P_{i2} , **. . . ,** P_{ik+1}) ; i = 0, 1, 2, **. . .** , M.

The above system of equations (2) is solved exploiting the special structure of the co-efficient matrix. It is solved using Numerical methods. Since there is no clear cut choice for M, we may start the iterative process by taking, say M = 1 and increase it until the individual elements of **X** do not change significantly. That is, if M^* denotes the truncation point then

$$\| \mathbf{x}^{\mathbf{M}^*}(\mathbf{i}) - \mathbf{x}^{\mathbf{M}^*-1}(\mathbf{i}) \|_\infty < \boldsymbol{\varepsilon}$$ where **ε** is an infinitesimal quantity.

4. STABILITY CONDITION

Theorem: The inequality $\left(\frac{\lambda}{\mu_1} + \frac{\lambda q_1}{\mu_2} + \frac{\lambda q_1 q_2}{\mu_3} + \cdots + \frac{\lambda q_1 q_2 \ldots q_{k-1}}{\mu_k} + \frac{\lambda p}{\mu} \right) < 1$ is the necessary and sufficient condition for system to be stable.

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial).
The stationary probability vector **X** satisfies

$$\mathbf{XQ = 0} \quad \text{and} \quad \mathbf{Xe = 1}$$
(3)

Let R be the rate matrix and satisfying the equation

$$\mathbf{A_0+RA_1+R^2 A_2 = 0}$$
(4)

The system is stable if sp(R) <1. According to Neuts **[13]**, the Matrix R satisfies sp(R) <1 if and only if

$$\mathbf{\Pi A_0 e < \Pi A_2 e}$$
(5)

where $\mathbf{\Pi} = (\pi_1,\ldots,\pi_{k+1})$ and satisfies

$$\mathbf{\Pi A = 0} \text{ and } \mathbf{\Pi e = 1}$$
(6)

and

$$\mathbf{A = A_0+A_1+A_2}$$
(7)

Here $\mathbf{A_0}$, $\mathbf{A_1}$ and $\mathbf{A_2}$ are square matrices of order k and $\mathbf{A_0} = \boldsymbol{\lambda} \mathbf{I}$. I is the corresponding Identity matrix,

$$\mathbf{A_1} = \begin{pmatrix} -(\lambda+\mu_1) & q_1\,\mu_1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -(\lambda+\mu_2) & q_2\,\mu_2 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -(\lambda+\mu_3) & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -(\lambda+\mu_{k-1}) & q_{k-1}\,\mu_{k-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & --(\lambda+\mu_k) & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\mu) \end{pmatrix}$$

$$\mathbf{A_2} = \begin{pmatrix} (1-p)(1-q_1)\mu_1 & 0 & 0 & \ldots & 0 & 0 & p(1-q_1)\mu_1 \\ (1-p)(1-q_2)\mu_2 & 0 & 0 & \ldots & 0 & 0 & p(1-q_2)\mu_2 \\ (1-p)(1-q_3)\mu_3 & 0 & 0 & \ldots & 0 & 0 & p(1-q_3)\mu_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (1-p)(1-q_{k-1})\mu_{k-1} & 0 & 0 & \ldots & 0 & 0 & p(1-q_{k-1})\mu_{k-1} \\ (1-p)\mu_k & 0 & 0 & \ldots & 0 & 0 & p\mu_k \\ \mu & 0 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$

By substituting $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$ in equations (5), (6) and (7) ,we get

$$\left(\frac{\lambda}{\mu_1}+\frac{\lambda q_1}{\mu_2}+\frac{\lambda q_1 q_2}{\mu_3}+\cdots+\frac{\lambda q_1 q_2 \ldots q_{k-1}}{\mu_k}+\frac{\lambda p}{\mu}\right)<1$$

The inequality $\left(\frac{\lambda}{\mu_1}+\frac{\lambda q_1}{\mu_2}+\frac{\lambda q_1 q_2}{\mu_3}+\cdots+\frac{\lambda q_1 q_2 \ldots q_{k-1}}{\mu_k}+\frac{\lambda p}{\mu}\right)<1$ is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after the departure of n^{th} customer from the service station. We first prove the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic if $\left(\frac{\lambda}{\mu_1}+\frac{\lambda q_1}{\mu_2}+\frac{\lambda q_1 q_2}{\mu_3}+\cdots+\frac{\lambda q_1 q_2 \ldots q_{k-1}}{\mu_k}+\frac{\lambda p}{\mu}\right)<1$. $\{Q_n, n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n, n \geq 0\}$ is positive recurrent.

According to the Foster criterion [**8**], the irreducible and aperiodic Markov chain $\{Q_n, n \geq 0\}$ is positive recurrent if $|\psi_m| < \infty$ for all m and $\lim_{m\to\infty} \sup \psi_m < 0$, where

$$\psi_m = E\left((Q_{n+1} - Q_n) / Q_n = m\right) \;;\quad m = 0, 1, 2, 3, \ldots$$

$$= \left(\frac{\lambda}{\mu_1}+\frac{\lambda q_1}{\mu_2}+\frac{\lambda q_1 q_2}{\mu_3}+\cdots+\frac{\lambda q_1 q_2 \ldots q_{k-1}}{\mu_k}+\frac{\lambda p}{\mu}\right)-\left(\frac{m\sigma}{\lambda+m\sigma}\right)$$

If $\left(\frac{\lambda}{\mu_1}+\frac{\lambda q_1}{\mu_2}+\frac{\lambda q_1 q_2}{\mu_3}+\cdots+\frac{\lambda q_1 q_2 \ldots q_{k-1}}{\mu_k}+\frac{\lambda p}{\mu}\right)<1$, then $|\psi_m| < \infty$ for all m and $\lim_{m\to\infty} \sup \psi_m < 0$

Therefore the embedded Markov chain $\{Q_n, n \geq 0\}$ is ergodic.

5. SPECIAL CASES

a. If $\sigma \to \infty$, then this model becomes single server classical queueing system with second optional service under Coxian phase type services.
b. If $q_1 = q_2 = q_3 = \ldots = q_{k-1} = 1$ and each $\mu_i = k\mu$ then this model becomes Single server retrial queueing system with second optional service under Erlang-k services discussed by Gopal sekar *et al*.,**[10]**.
c. If $p \to 0$, then this model becomes single server Retrial queueing system with Coxian phase type services.

d. If $q_1 = q_2 = q_3 = \ldots = q_{k-1} = 1$, $p \to 0$ and each $\mu_i = k\mu$ then this model becomes Single server retrial queueing system with Erlang-k services discussed by Ayyappan *et al.*,**[5].**

6. SYSTEM PERFORMANCE MEASURES

In this section some important performance measures along with formulas and their qualitative behaviour for this queueing model are studied. Numerical study has been dealt in very large scale to study these measures. Defining

$P(n, 0)$ = Probability that there are n customers in the orbit and server is idle

$P(n, j)$ = Probability that there are n customers in the orbit and server is busy with a customer in the j^{th} phase (j = 1,2,3, . . . ,k)

$P(n, k+1)$ = Probability that there are n customers in the orbit and server is busy with a customer who opt for optional service.

We can find various probabilities for various values of λ, $q_1, q_2, \ldots q_{k-1}$, μ_1, μ_2, μ_3, . . . μ_k, k, μ, p and σ from section 3 and the following system measures can be studied with these probabilities.

a. The probability mass function of server state

Prob (the server is idle) = $\sum_{i=0}^{\infty} p(i,0)$

Prob (the server is busy with customer in the j^{th} phase) = $\sum_{i=0}^{\infty} p(i,j)$; j = 1, 2,3,. . . , k

Prob (the server is busy with customer who opted for optional service) = $\sum_{i=0}^{\infty} p(i,k+1)$

b. The probability mass function of number of customers in the orbit

Prob (no customers in the orbit) = $\sum_{j=0}^{k+1} p(0,j)$

Prob (i customers in the orbit) = $\sum_{j=0}^{k+1} p(i,j)$

c. The mean number of customers in the orbit(MNCO)

$$= \sum_{i=0}^{\infty} i \left(\sum_{j=0}^{k+1} p(i,j) \right)$$

d. The probability that the orbiting customer is blocked

Blocking Probability = $\sum_{i=1}^{\infty} \sum_{j=1}^{k+1} p(i,j)$

e. The probability that an arriving customer enter into service immediately

$$= \sum_{i=0}^{\infty} p(i,0)$$

7. NUMERICAL STUDY

The values of parameters λ, q_1, q_2 ,. . . q_{k-1}, μ_1, μ_2 , μ_3, . . . μ_k, k, μ, p and σ are chosen so that they satisfy the stability condition discussed in section 4. The system performance measures of this model have been done and expressed in the form of tables which are shown below using the steady state probability vector **X** for various values of λ, q_1,q_2,. . . q_{k-1}, μ_1, μ_2 , μ_3, . . . μ_k, k, p and σ.

For $\lambda = 8$,$\mu_1 = 30$, $\mu_2 = 20$, $\mu_3 =15$, $\mu_4 = 30$, $\mu_5 = 25$, $q_1 = 0.6$, $q_2 = 0.8$, $q_3 = 0.4$, $q_4 = 0.2$, p = 0.8 , $\mu = 10$ and $\sigma = 100$, the steady state probability vector is **X** = (x[0] , x[1], x[2] , **. . .** , x[M])

x(0) = [0.1280 0.0500 0.0321 0.0223 0.0035 0.0006 0.0007]
x(1) = [0.0087 0.0339 0.0310 0.0293 0.0054 0.0011 0.0016]
x(2) = [0.0041 0.0272 0.0263 0.0285 0.0056 0.0013 0.0021]
x(3) = [0.0024 0.0229 0.0223 0.0254 0.0052 0.0013 0.0024]
x(4) = [0.0016 0.0196 0.0190 0.0220 0.0046 0.0011 0.0023]
x(5) = [0.0011 0.0168 0.0162 0.0189 0.0040 0.0010 0.0021]
x(6) = [0.0008 0.0143 0.0139 0.0162 0.0034 0.0009 0.0019]
x(7) = [0.0006 0.0122 0.0118 0.0139 0.0029 0.0007 0.0017]
x(8) = [0.0004 0.0104 0.0101 0.0119 0.0025 0.0006 0.0014]
x(9) = [0.0003 0.0089 0.0086 0.0101 0.0021 0.0005 0.0012]
x(10) = [0.0003 0.0076 0.0073 0.0086 0.0018 0.0005 0.0011]
x(11) = [0.0002 0.0065 0.0062 0.0073 0.0015 0.0004 0.0009]
x(12) = [0.0002 0.0055 0.0053 0.0062 0.0013 0.0003 0.0008]
x(13) = [0.0001 0.0047 0.0045 0.0053 0.0011 0.0003 0.0007]

Similarly, we can find x (n) for $n \geq 14$ and it is noticed that $x(n) \rightarrow 0$ as $n \rightarrow \infty$. For the numerical parameters chosen above, $x(n) \rightarrow 0$ for $n \geq 84$ and the sum of the steady state probabilities becomes 0.99999999999. In the same manner, we can find steady state probability vector **X** for all values of λ, q_1,q_2,. . . q_{k-1}, μ_1, μ_2 , μ_3, . . . μ_k, k, μ, p and σ . By using these steady state probability vector , we can find the following system measures

1. Probability that the server is idle = 0.149269
2. Probability that the server is busy with a customer in phase 1 = 0.266667
   Probability that the server is busy with a customer in phase 2 = 0.240000
   Probability that the server is busy with a customer in phase 3 = 0.256000
   Probability that the server is busy with a customer in phase 4 = 0.051200
   Probability that the server is busy with a customer in phase 5 = 0.012288
3. Probability that the server is busy with a regular service = 0.826155
4. probability that the server is busy with a customer who opt for optional service
   = 0.024576
5. Probability mass function of number of customers in the orbit

No. of customers in the orbit	Probability	No. of customers in the orbit	Probability	No. of customers in the orbit	Probability
0	0.237292	7	0.043839	14	0.014175
1	0.110948	8	0.037381	15	0.012041
2	0.095089	9	0.031848	16	0.010226
3	0.081834	10	0.027115	17	0.008682
4	0.070214	11	0.023072	18	0.007369
5	0.060103	12	0.019622	19	0.006254
6	0.051363	13	0.016681	20	0.005306

6. Mean number of customers in the orbit = 5.138255
7. Probability that the orbiting customer is blocked = 0.741438

Table 1 shows the impact of low arrival rate and retrial rate σ over Mean number of customers in the orbit and we infer the following

- Mean number of customers in the orbit decreases as retrial rate σ increases.
- Mean number of customers in the orbit increases as λ increases.
- $P_0 = 0.7896$, $P_1 = 0.2065$ and $P_2 = 0.0038$ are independent of retrial rate σ.
- This model becomes a classical queueing model with second optional service under Coxian type services if σ is large

Table 1 : Low arrival rate against Mean number of customers in the orbit for $\lambda = 2$
$\mu_1 = 30$, $\mu_2 = 20$, $\mu_3 =15$, $\mu_4 = 30$, $\mu_5 = 25$, $q_1 = 0.6$, $q_2 = 0.8$, $q_3 = 0.4$, $\mu = 10$
$q_4 = 0.2$, $p = 0.5$ and various values of σ

σ	OCUT	MNCO
10	8	0.1067
20	8	0.0801
30	8	0.0712
40	8	0.0668
50	8	0.0641
60	8	0.0623
70	8	0.0610
80	8	0.0601
90	8	0.0594
100	8	0.0588
200	8	0.0561
300	8	0.0552
400	8	0.0548
500	8	0.0545
600	8	0.0543
700	8	0.0542

800	8	0.0541
900	8	0.0540
1000	8	0.0540
2000	8	0.0537
3000	8	0.0536
4000	8	0.0536
5000	8	0.0535
6000	8	0.0535
7000	8	0.0535
8000	8	0.0535
9000	8	0.0535
10000	8	0.0535

MNCO: Mean number of customers in the orbit; P_0: Probability that the server is idle
P_1: Probability that the server is busy with regular service; P_2: Probability that the server is busy with optional service; σ: Retrial rate; OCUT: Truncation level

Table 2 shows the impact of medium arrival rate and retrial rate σ over Mean number of customers in the orbit and we infer the following

- Mean number of customers in the orbit decreases as retrial rate σ increases.
- Mean number of customers in the orbit increases as λ increases.
- $P_0 = 0.4741$, $P_1 = 0.5163$ and $P_2 = 0.0096$ are independent of retrial rate σ.
- This model becomes a classical queueing model with second optional service under Coxian type services if σ is large

Table 2 : Medium arrival rate against Mean number of customers in the orbit $\lambda = 5$, $\mu_1 = 30$, $\mu_2 = 20$, $\mu_3 = 15$, $\mu_4 = 30$, $\mu_5 = 25$, $q_1 = 0.6$, $q_2 = 0.8$, $q_3 = 0.4$, $\mu = 10$ $q_4 = 0.2$, $p = 0.5$ and various values of σ

σ	OCUT	MNCO
10	22	1.1110
20	21	0.8336
30	21	0.7412
40	20	0.6950
50	20	0.6672
60	20	0.6487
70	20	0.6355
80	20	0.6256
90	20	0.6179
100	20	0.6117
200	20	0.5840
300	20	0.5748
400	20	0.5701
500	20	0.5674
600	20	0.5655
700	20	0.5642

800	20	0.5632
900	20	0.5624
1000	20	0.5618
2000	20	0.5590
3000	20	0.5581
4000	20	0.5577
5000	20	0.5574
6000	20	0.5572
7000	20	0.5571
8000	20	0.5570
9000	20	0.5569
10000	20	0.5568

Table 3 shows the impact of high arrival rate λ and retrial rate σ over Mean number of customers in the orbit and we infer the following

- Mean number of customers in the orbit decreases as retrial rate σ increases.
- Mean number of customers in the orbit increases as λ increases.
- $P_0 = 0.1585$, $P_1 = 0.8262$ and $P_2 = 0.0154$ are independent of retrial rate σ.
- This model becomes a classical queueing model with second optional service under Coxian type services if σ is large

Table 3 : High arrival rate against Mean number of customers in the orbit $\lambda = 8$, $\mu_1 = 30$, $\mu_2 = 20$, $\mu_3 = 15$, $\mu_4 = 30$, $\mu_5 = 25$, $q_1 = 0.6$, $q_2 = 0.8$, $q_3 = 0.4$, $\mu = 10$, $q_4 = 0.2$, $p = 0.5$ and various values of σ

σ	OCUT	MNCO
10	90	8.5074
20	84	6.3835
30	81	5.6755
40	80	5.3215
50	80	5.1091
60	79	4.9676
70	79	4.8664
80	78	4.7906
90	78	4.7316
100	78	4.6844
200	77	4.4720
300	77	4.4012
400	77	4.3658
500	77	4.3445
600	77	4.3304

700	77	4.3203
800	77	4.3127
900	77	4.3068
1000	77	4.3021
2000	77	4.2808
3000	77	4.2738
4000	77	4.2702
5000	77	4.2681
6000	77	4.2667
7000	77	4.2657
8000	77	4.2649
9000	77	4.2643
10000	77	4.2638

Table 4 shows the impact of p over Mean number of customers in the orbit and we infer the following

- Mean number of customers in the orbit decreases as p decreases.
- $P_1 = 0.6196$ is independent of retrial rate σ.
- This model becomes a Single server Retrial queueing model with Coxian type services if $p \to 0$.

Table 4: P against Mean number of customers in the orbit for $\lambda = 6$, $\mu_1 = 30$, $\mu_2 = 20$, $\mu_3=15$, $\mu_4 = 30$, $\mu_5 = 25$, $q_1 = 0.6$, $q_2 = 0.8$, $q_3 = 0.4$, $\mu = 10$, $q_4 = 0.2$, $p = 0.8$ and $\sigma = 10$

p	OCUT	MNCO	P_0	P_2
0.80000	33	2.1438	0.3620	0.0184
0.40000	31	2.0275	0.3712	0.0092
0.20000	31	1.9715	0.3758	0.0046
0.10000	30	1.9440	0.3781	0.0023
0.05000	30	1.9304	0.3792	0.0012
0.02500	30	1.9236	0.3798	0.0006
0.01250	30	1.9203	0.3801	0.0003
0.00625	30	1.9186	0.3802	0.0001
0.00313	30	1.9177	0.3803	0.0001
0.00156	30	1.9173	0.3803	0.0000
0.00078	30	1.9171	0.3804	0.0000
0.00039	30	1.9170	0.3804	0.0000
0.00020	30	1.9169	0.3804	0.0000
0.00010	30	1.9169	0.3804	0.0000
0.00005	30	1.9169	0.3804	0.0000
0.00002	30	1.9169	0.3804	0.0000
0.00001	30	1.9169	0.3804	0.0000
0.00001	30	1.9169	0.3804	0.0000
0.00000	30	1.9169	0.3804	0.0000

0.00000	30	1.9169	0.3804	0.0000

p : probability that the customer opts an optional service

8. CONCLUSION

The Numerical study shows the changes in the system due to impact of retrial rate. The mean number of customers in the orbit decreases as retrial rate increases and it increases as arrival rate increases. As p decreases, the mean number of customers in the orbit decreases and this model becomes Single server Retrial queueing system with Coxian phase type services. The various special cases have been discussed and which are particular cases of this research work. This research work will be extended further by introducing various vacation policies, negative arrival and unreliable server.

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